Int J Theor Phys (2008) 47: 3310–3320
DOI 10.1007/s10773-008-9766-2

Can Sobolev Inequality Be Written for Sharma-Mittal Entropy?

O. Üzengi Aktürk · E. Aktürk · M. Tomak

Received: 19 March 2008 / Accepted: 22 May 2008 / Published online: 4 June 2008

Abstract In this paper, we focus on Sobolev inequality in the context of Sharma-Mittal entropy. Using this new inequality, generalized entropic uncertainty relation in accordance with Sharma-Mittal entropy is derived and the pseudoadditivity relation has been obtained. This new entropic uncertainty relation has then been applied to physical examples such as one dimensional harmonic oscillator and Pösch-Teller potential. Finally, it has been shown that for certain values of the parameters of Sharma-Mittal measure, the present results reduce to the corresponding results of Shannon, Renyi and Tsallis measures.

Keywords Sharma-Mittal entropy · Sobolev inequality · Tsallis entropy · Renyi entropy · Shannon entropy

1 Introduction

Recently, there has been an increase in the interest to generalize Boltzmann-Gibbs entropy by some other entropies such as Tsallis [1] and Rényi [2] entropies. Tsallis entropy has been applied to many systems ranging from nonlinear diffusion equations [3], Fokker-Planck systems [4, 5], the specific heat of the harmonic oscillator [6], one-dimensional Ising model [7, 8], Boltzmann H-theorem [9–11], Ehrenfest theorem [12], quantum statistics [13, 14], paramagnetic systems [15], Circle maps [16], Henon map [17], Haldane exclusion statistics [18], q-expectation value [19] to quantum information [20]. Rényi entropy has been studied as a generalization of Boltzmann-Gibbs entropy since it too yields inverse power law

O. Üzengi Aktürk (✉) · M. Tomak
Department of Physics, Middle East Technical University, 06531 Ankara, Turkey
e-mail: uolcay@metu.edu.tr

M. Tomak
e-mail: tomak@metu.edu.tr

E. Aktürk
Department of Physics, Hacettepe University, 06800 Ankara, Turkey
e-mail: eakturk@hacettepe.edu.tr

distributions when maximized under constraints [21] and has been shown to satisfy the zeroth law of thermodynamics [22]. Later, a new entropy measure has been not introduced, which appears to generalize both of these entropies [23–25]. This is the two-parameter Sharma-Mittal entropy [26]. The properties of Sharma-Mittal entropy has been investigated by some authors [23, 24, 27]. It generalizes Tsallis, Rényi and Boltzmann-Gibbs (BG) entropies through the manipulation of its two parameters thereby reducing to these entropies as limiting cases.

A new uncertainty relation in quantum mechanics is derived in [28] regarding the Shannon-von Neumann entropies mixed with the momentum and coordinate probability densities by employing the Sobolev inequality. The investigation of various properties of the Sobolev inequality is given in [29]. Information entropy of classical orthogonal polynomials and their application to the harmonic oscillator and Coulomb potentials is investigated in [30, 31]. The Sobolev inequality have been studied by using Tsallis [32] and Renyi entropies [33]. However, in the literature, the Sobolev inequality has not been investigated by using the Sharma-Mittal entropy. The generalization of Sobolev inequality in this particular context is important since it will include all the known inequalities such as the ones derived from Shannon, Tsallis or Rényi entropies. Moreover, the uncertainty in position observable can be divergent for the power-law quantum wave packet in the configuration space. Hence, the Heisenberg type uncertainty will not have any applicability. Another example for which ordinary form of Heisenberg uncertainty relation fails to work is encountered in the example of quantum damped oscillator. In this particular example, solutions provided in the literature generally suffer from disappearing asymptotically in time [34]. On the other hand, the information entropic uncertainty relation includes only finite quantities. Thus it is more suitable than Heisenberg type uncertainty relation [35]. To demonstrate this generalized picture, the information properties in position and momentum space for D-dimensional harmonic oscillator system are studied using Sharma-Mittal entropy. These mathematical expressions then have been applied to physical examples such as one dimensional harmonic oscillator and Pösch-Teller potential. Finally, it has been shown that for certain values of the parameters of Sharma-Mittal measure, the present results reduce to the corresponding results of Shannon, Renyi and Tsallis measures.

This paper is organized as follows. In Sect. 2, we provide the rudiments needed for the generalization of Sobolev inequality. In Sect. 3, we calculate the position and momentum information entropies analytically for D-dimensional harmonic oscillator. To verify this result, same process is used for one dimensional harmonic oscillator and Pösch-Teller potential in Sect. 4. Finally, we present the conclusion in Sect. 5.

2 Sharma-Mittal Entropy

The Sharma-Mittal entropy is given by

$$S_{SM}(p) = \frac{1}{1-s}\Big[\Big(\sum_i p_i^q\Big)^{(\frac{1-s}{1-q})} - 1\Big]. \tag{1}$$

In the limit $s \to 1$, Sharma-Mittal entropy becomes Rényi entropy [2] which is

$$S_R(p) = \frac{1}{1-q}\ln\Big(\sum_i p_i^q\Big), \tag{2}$$

for $s \to q$, it is Tsallis entropy [1] given by

$$S_T(p) = \frac{\sum_i p_i^q - 1}{1-q}. \tag{3}$$

In the limiting case when both parameters approach one, we recover the ordinary Boltzmann-Gibbs (BG) entropy which reads

$$S_{BG}(p) = -\sum_i p_i \ln p_i. \tag{4}$$

we can write (1) as,

$$S_{SM\gamma}(q) = \frac{1}{1-s}\Big[\Big(\int d^D r (|\gamma(r)|^\alpha\Big)^{(\frac{1-s}{1-\alpha})} - 1\Big], \tag{5}$$

and

$$S_{SM\nu}(p) = \frac{1}{1-s}\Big[\Big(\int d^D k (|\nu(k)|^\beta\Big)^{(\frac{1-s}{1-\beta})} - 1\Big], \tag{6}$$

for the positions and momentum space entropies, respectively, where $\gamma(r) = |\psi(r)|^2$ and $\nu(k) = |\tilde{\psi}(k)|^2$ are quantum mechanical probability densities in r- and k-space, respectively. Here $\tilde{\psi}(k)$ is the Fourier transform of $\psi(r)$.

For two statistically independent systems given by probability distributions q and p, Sharma-Mittal entropy satisfies [24]

$$S_{SM}(p+q) = S_{SM}(p) + S_{SM}(q) + (1-s)S_{SM}(p)S_{SM}(q). \tag{7}$$

This relation is important in order to see how Sharma-Mittal measure includes both Tsallis and Rényi entropies as its limiting cases. When we take $s = q$, it becomes the relation satisfied by Tsallis entropy and shows its nonadditivity. Moreover, when $s = 1$, we have the additive property which relates to Rényi entropy. In this sense Sharma-Mittal entropy includes both additive and nonadditive features in it.

3 Entropic Uncertainty of Sharma-Mittal Entropy

In this section, we consider a D-dimensional harmonic oscillator for the investigation of entropic uncertainty of Sharma-Mittal entropy. The normalized eigenfunctions of the D-dimensional harmonic oscillator in position and momentum space are given by [31]. Using the (5) and wave function, the position-space information integral in the generalized form for the D-dimensional harmonic oscillator can be obtained as

$$S_{SM\gamma}(q) = \frac{1}{1-s}\Big[\Big\{\Big(\frac{2n!\lambda^{l+D/2}}{\Gamma(n+l+D/2)}\Big)^\alpha I_1 I_2\Big\}^{\frac{1-s}{1-\alpha}} - 1\Big], \tag{8}$$

where the integrals I_1 and I_2 are given by

$$I_1 = \int r^{2l\alpha+D-1} e^{-\lambda r^2 \alpha} [L_n^{l+D/2-1}(\lambda r^2)]^{2\alpha} dr, \tag{9}$$

and

$$I_2=\int [Y_{l,m}(\Omega_D)]^{2\alpha} d\Omega_D, \tag{10}$$

I_1 and I_2 are the contributions to generalized entropies from the Laguerre polynomial and hyperspherical harmonics, respectively. In momentum space we can write the generalized entropy as

$$S_{SM\nu}(p)=\frac{1}{1-s}\Big[\Big\{\Big(\frac{2n!\lambda^{-l-D/2}}{\Gamma(n+l+D/2)}\Big)^{\beta} I_3 I_2\Big\}^{\frac{1-s}{1-\beta}}-1\Big], \tag{11}$$

where the I_3 is given by

$$I_3=\int k^{2l\beta+D-1} e^{-k^2\beta/\lambda}[L_n^{l+D/2-1}(k^2/\lambda)]^{2\beta} dk. \tag{12}$$

For the ground state, $n=0$, we can write in position space

$$S_{SM\gamma}(q)=\frac{1}{1-s}\Big[\Big\{\Big(\frac{\lambda}{\pi}\Big)^{D(\alpha-1)/2}\frac{1}{\alpha^{D/2}}\Big\}^{\frac{1-s}{1-\alpha}}-1\Big]. \tag{13}$$

If this equation is rewritten, we obtain

$$(1+(1-s)S_{SM\gamma}(q))^{\frac{1-\alpha}{(1-s)2\alpha}}=\Big\{\Big(\frac{\lambda}{\pi}\Big)^{D(\alpha-1)/4\alpha}\frac{1}{\alpha^{D/4\alpha}}\Big\}, \tag{14}$$

and in momentum space

$$S_{SM\nu}(p)=\frac{1}{1-s}\Big[\Big\{\Big(\frac{1}{\lambda\pi}\Big)^{D(\beta-1)/2}\frac{1}{\beta^{D/2}}\Big\}^{\frac{1-s}{1-\beta}}-1\Big], \tag{15}$$

when (15) is rearranged, we find the following form

$$(1+(1-s)S_{SM\nu}(p))^{\frac{1-\beta}{(1-s)2\beta}}=\Big\{\Big(\frac{1}{\lambda\pi}\Big)^{D(\beta-1)/4\beta}\frac{1}{\beta^{D/4\beta}}\Big\}. \tag{16}$$

We can write for the ratio (16) to (14) as,

$$\frac{(1+(1-s)S_{SM\nu}(p))^{\frac{1-\beta}{(1-s)2\beta}}}{(1+(1-s)S_{SM\gamma}(q))^{\frac{1-\alpha}{(1-s)2\alpha}}}=\lambda^{\frac{D}{4\beta}+\frac{D}{4\alpha}-\frac{D}{2}}\Big(\frac{\pi}{\alpha}\Big)^{(-D/4\alpha)}\Big(\frac{\pi}{\beta}\Big)^{(D/4\beta)}. \tag{17}$$

If the power terms of λ is equal to zero, we obtain Sobolev conditions as

$$\frac{1}{\beta}+\frac{1}{\alpha}=2. \tag{18}$$

As can be seen from (17), when Sobolev condition is satisfied, the ratio is equality and if not, inequality. Hence, we find the entropic uncertainty relation based on the Sobolev inequality as

$$\frac{[1+(1-s)S_{SM\nu}(p)]^{\frac{1-\beta}{(1-s)2\beta}}}{[1+(1-s)S_{SM\gamma}(q)]^{\frac{1-\alpha}{(1-s)2\alpha}}}\le\Big(\frac{\pi}{\alpha}\Big)^{(-D/4\alpha)}\Big(\frac{\pi}{\beta}\Big)^{(D/4\beta)}, \tag{19}$$

with the condition $\frac{1}{\beta}+\frac{1}{\alpha}=2$. Moreover, using the (15) and (13) the pseudo additivity properties are also checked from (7). If we take $s=\alpha$ and $s=\beta$, coordinate and momentum space, respectively, the new Sobolev inequality becomes the relation satisfied by Tsallis entropy [32] as

$$\frac{[1+(1-\beta)S_{T\nu}(p)]^{\frac{1}{2\beta}}}{[1+(1-\alpha)S_{T\gamma}(q)]^{\frac{1}{2\alpha}}}\le\left(\frac{\pi}{\alpha}\right)^{(-D/4\alpha)}\left(\frac{\pi}{\beta}\right)^{(D/4\beta)}, \tag{20}$$

with the condition $\frac{1}{\beta}+\frac{1}{\alpha}=2$. Moreover, when $s=1$, we have the extensive property which relates to Rényi entropy

$$\frac{[e^{(1-\beta)S_{R\nu}(p)}]^{\frac{1}{2\beta}}}{[e^{(1-\alpha)S_{R\gamma}(q)}]^{\frac{1}{2\alpha}}}\le\left(\frac{\pi}{\alpha}\right)^{(-D/4\alpha)}\left(\frac{\pi}{\beta}\right)^{(D/4\beta)}, \tag{21}$$

with the condition $\frac{1}{\beta}+\frac{1}{\alpha}=2$. Writing (21) in a new form, one obtains [33]

$$S_{R\nu}(p)+S_{R\gamma}(q)\ge-\frac{D}{2}\left(\frac{\ln\alpha}{1-\alpha}+\frac{\ln\beta}{1-\beta}\right)+\ln(\pi^D). \tag{22}$$

In the limit of α, β going to unity, we obtain conventional entropic uncertainty relation

$$D(1+\ln\pi)\le S_{BG}(p)+S_{BG}(q), \tag{23}$$

where $S_{BG}(p)$ and $S_{BG}(q)$ are the usual Shannon-von Neumann entropies associated with position and momentum. In case of $D=1$, the entropic uncertainty relation is reduced to $1+\ln\pi$.

4 Applications

4.1 One Dimensional Simple Harmonic Oscillator (SHO)

One dimensional simple harmonic oscillator wave functions are given by

$$\psi_n(x)=\left(\frac{1}{2^n n!\sqrt{\pi}}\right)^{\frac{1}{2}}\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}}H_n\left(\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}x\right)\exp\left[-\frac{m\omega}{2\hbar}x^2\right]. \tag{24}$$

For the ground state, (5) can be rewritten as

$$\Big(1+(1-s)S_{SM\gamma}(q)\Big)^{\frac{1-\alpha}{1-s}}=\int(|\psi_0|^2)^\alpha dx, \tag{25}$$

and hence we obtain the relation,

$$\Big(1+(1-s)S_{SM\gamma}(q)\Big)^{\frac{1-\alpha}{(1-s)2\alpha}}=\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{\alpha-1}{4\alpha}}\frac{1}{\alpha^{1/4\alpha}}. \tag{26}$$

We show our result for the coordinate space Sharma-Mittal entropy in Fig. 1. In this figure and hereafter, $\hbar,\omega$ and m are set equal to unity for simplicity. As seen from the figure, $S_{SM\gamma}(q)$ is large for small α and s but this entropy monotonically decreases for larger α

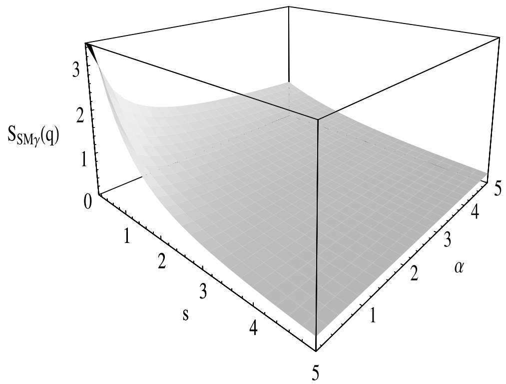

Fig. 1 $S_{SM\gamma}(q)$ as a function of α and s in arbitrary units

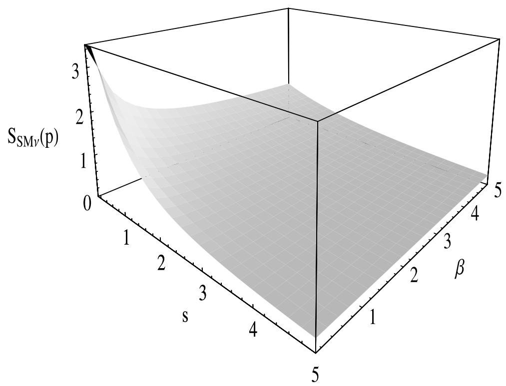

Fig. 2 $S_{SM\nu}(p)$ as a function of β and s in arbitrary units

and s. Larger values of these parameters give rise to a smaller value of entropy and therefore the wave packet is more localized. On the other hand, the momentum wave function is

$$\tilde{\psi}_n(p) = \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} \left(\frac{1}{m\omega\hbar}\right)^{\frac{1}{4}} H_n\left(\left(\frac{1}{m\omega\hbar}\right)^{\frac{1}{2}} p\right) \exp\left[-\frac{1}{2m\omega\hbar} p^2\right]. \tag{27}$$

For the ground state in momentum space, if the (6) is rearranged to give

$$\left(1 + (1-s) S_{SM\nu}(p)\right)^{\frac{1-\beta}{1-s}} = \int (|\tilde{\psi}_0|^2)^{\beta} dp, \tag{28}$$

and we obtain,

$$\left(1 + (1-s) S_{SM\nu}(p)\right)^{\frac{1-\beta}{(1-s)2\beta}} = \left(\frac{1}{m\omega\hbar\pi}\right)^{\frac{\beta-1}{4\beta}} \frac{1}{\beta^{1/4\beta}}. \tag{29}$$

The momentum space Sharma-Mittal entropy is shown in Fig. 2. As can be seen from the figure, this entropy shows the same behavior as in Fig. 1. Choosing $\hbar = 1$ and dividing (28) by (25), we have

$$\frac{(1 + (1-s) S_{SM\nu}(p))^{\frac{1-\beta}{(1-s)2\beta)}}}{(1 + (1-s) S_{SM\gamma}(q))^{\frac{1-\alpha}{(1-s)2\alpha}}} = (m\omega)^{\frac{1}{4\alpha} + \frac{1}{4\beta} - \frac{1}{2}} \left(\frac{\pi}{\beta}\right)^{\frac{1}{4\beta}} \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{4\alpha}}. \tag{30}$$

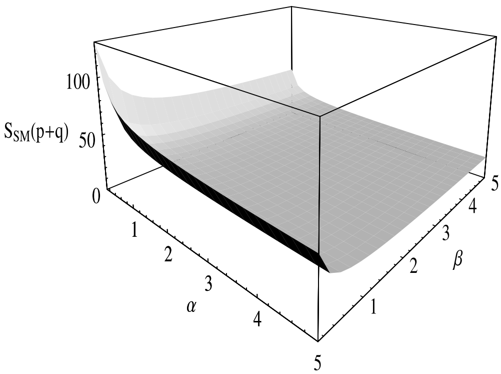

Fig. 3 $S_{SM}(p+q)$ as a function of β and α for $s=0.1$ in arbitrary units

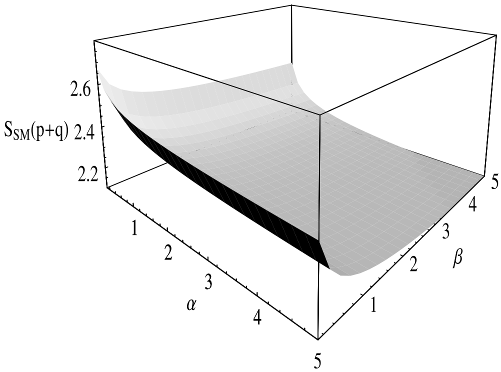

Fig. 4 $S_{SM}(p+q)$ as a function of β and α for $s=1.3$ in arbitrary units

Taking the power of $m\omega$ to zero, the Sobolev condition is obtained as,

$$\frac{1}{\alpha}+\frac{1}{\beta}=2. \tag{31}$$

Using this condition, we obtain Sobolev inequality as

$$\frac{(1+(1-s)S_{SM\nu}(p))^{\frac{1-\beta}{(1-s)2\beta}}}{(1+(1-s)S_{SM\gamma}(q))^{\frac{1-\alpha}{(1-s)2\alpha}}}\leq\left(\frac{\pi}{\beta}\right)^{\frac{1}{4\beta}}\left(\frac{\pi}{\alpha}\right)^{-\frac{1}{4\alpha}}. \tag{32}$$

This result is in agreement with (19) for $D=1$. The plot of total Sharma-Mittal entropy using (7) is shown in Figs. 3 and 4 for small and large values of s, respectively. As seen from Figs. 3 and 4, the total Sharma-Mittal entropy of this system is monotonically decreasing. It has same form for small and large value of s whereas its magnitudes are different. On the other hand, in the limiting case when both parameters approach one, we also obtain the total entropy as equal to the exact lower bound $1+\ln\pi$.

4.2 Pösch-Teller Systems

The ground state wave function for hyperbolic PÖSCHL-TELLER potential is given by [36]

$$\psi_n^0(x)=\frac{1}{\sqrt{2B(\frac{1}{2},n)}}.\text{sech}^n\left(\frac{x}{2}\right), \tag{33}$$

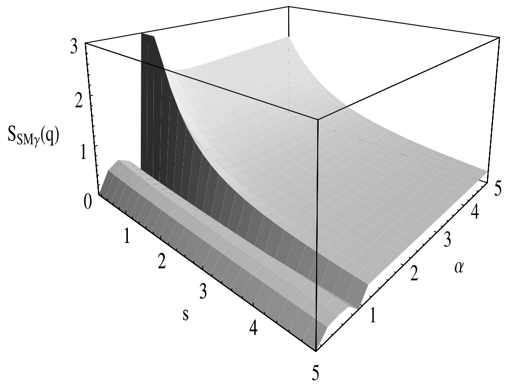

Fig. 5 $S_{SM\gamma}(q)$ as a function of α and s in arbitrary units

where $B(\frac{1}{2}, n)$ is the Beta-function. Using (5) and ground state wave function for $n = 1$, the position space entropy becomes,

$$\left(1 + (1-s)S_{SM\gamma}(q)\right)^{\frac{1-\alpha}{(1-s)}} = \frac{2^{1-2\alpha}\sqrt{\pi}\,\Gamma(\alpha)}{\Gamma(\alpha+\frac{1}{2})}, \tag{34}$$

where $\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}$ is given by

$$\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} = \sqrt{\alpha}\left(1 - \frac{1}{8\alpha} + \frac{1}{128\alpha^2} + \dots\right) \approx \sqrt{\alpha}. \tag{35}$$

Inserting this into (33) yields,

$$\left(1 + (1-s)S_{SM\gamma}(q)\right)^{\frac{1-\alpha}{(1-s)2\alpha}} = 2^{\frac{1-2\alpha}{2\alpha}}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{4\alpha}}. \tag{36}$$

The coordinate space Sharma-Mitall entropy is shown in Fig. 5. As seen from the figure, Sharma-Mittal entropy for coordinate space monotonically decreases with larger values while it shows fluctuations for smaller values of α and s. The momentum ground state wave function for $n = 1$ is given by

$$\psi_0(p) = \sqrt{\frac{\pi}{2}}\operatorname{sech}(\pi p). \tag{37}$$

Using (6) and momentum wave function, we find

$$\left(1 + (1-s)S_{SM\nu}(p)\right)^{\frac{1-\beta}{(1-s)2\beta}} = 2^{-\frac{1}{2}}\pi^{-\frac{1}{4\beta}+\frac{1}{2}}\left(\frac{1}{\beta}\right)^{\frac{1}{4\beta}}. \tag{38}$$

In Fig. 6, we present a plot of momentum space Sharma-Mittal entropy. Fig. 6 indicates that entropy is very small decreasing smaller values of β and s to β and s are unity but it is monotonically decreasing for larger values. When we can write for ratio (37) to (35), it is easier to obtain these ratios as (31) with Sobolev conditions. The behavior of total Sharma-Mittal entropy using the (7) is shown in Figs. 7 and 8 for value of α and β at $s = 0.1$ and $s = 1.3$, respectively.

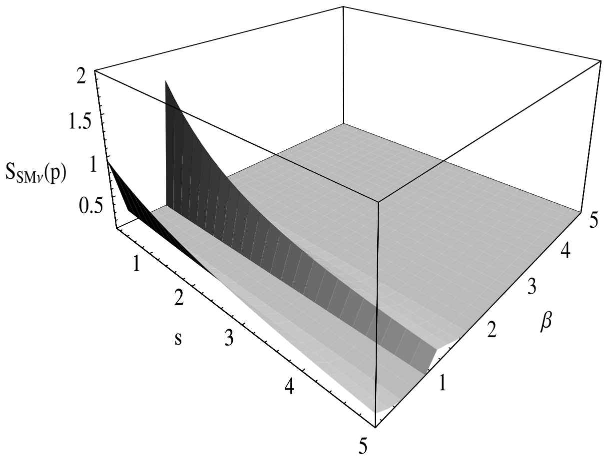

Fig. 6 $S_{SM\nu}(p)$ as a function of β and s in arbitrary units

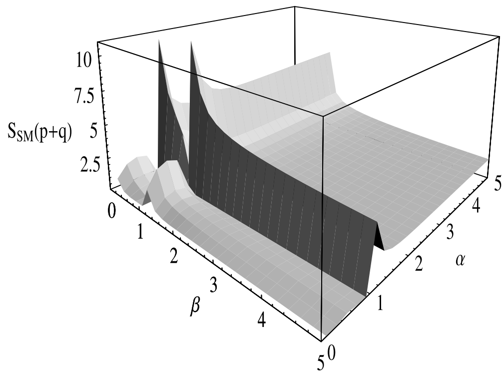

Fig. 7 $S_{SM}(p+q)$ as a function of β and α at $s = 0.1$ in arbitrary units

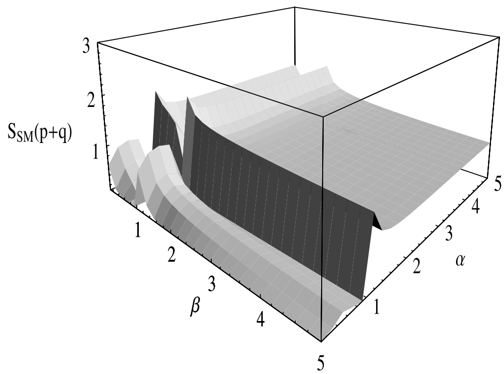

Fig. 8 Plot of $S_{SM}(p+q)$ with respect to β and α at $s = 1.3$ in arbitrary units

As seen from Figs. 7 and 8, the total Sharma-Mittal entropy of this system is monotonically decreasing for the value of α which is greater than unity. It has the same form for small and large value of s whereas its magnitudes are different.

5 Conclusion

Information entropic uncertainty relations is found to be useful in some particular cases in which ordinary Heisenberg uncertainty relation diverges or becomes zero as time increases. These situations are faced when one encounters for example power law wave packets [35] or damped quantum oscillators [34] for example. In order to remedy this, entropic uncertainty relations based on different entropies such as Shannon, Tsallis [32] or Renyi entropies [33] have been derived and used in applications. Recently, it has been found out that the Sharma-Mittal entropy generalizes Shannon, Rényi and Tsallis entropies through the adjustment of its two parameters. In this sense, it is the most general entropy measure in the literature so far. In order to have the most general entropic uncertainty relation, we have derived the Sobolev inequality in accordance with this measure and applied it to D-dimensional harmonic oscillator and hyperbolic Pösch-Teller potential. We have examined whether this explanation is reasonable from the entropic uncertainty point of view using the associated Sharma-Mittal entropy. We have found that the momentum and coordinate space Sharma-Mittal entropy of quantum mechanical systems monotonically decrease with respect to large values of β and α.

For the SHO case in both momentum and coordinate space Sharma-Mittal entropy has a decreasing behavior whereas it has different one in hyperbolic Pösch-Teller potential for β and α which are smaller than unity. In addition, the overall form of $S_{SM}(p+q)$ as a function of β and α which are larger than unity does not change. According to Sharma-Mittal inequality (32), the ratio can not be decreased beyond a certain value.

The maximum value of Sharma-Mittal uncertainty occurs at β and α equal to unity. We believe the Sharma-Mittal uncertainty inequality can be of help in further theoretical investigations where ordinary Heisenberg uncertainty relation diverges or goes to zero by time in cases such as inverse power law packets or quantum damped oscillator.

Acknowledgements This work was supported by Graduate School of Natural and Applied Sciences of METU (No: BAP-2006-07.02-00-01). Fruitful discussions with G.B. Bağcı is gratefully acknowledged.

References

1. Tsallis, C.: J. Stat. Phys. **52**, 479 (1988)
2. Rényi, A.: On measures of entropy and information. In: Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, vol. 1, pp. 547–561. University California Press, Berkeley (1961)
3. Plastino, A.R., Casas, M., Plastino, A.: Phys. A **280**, 289 (2000)
4. Stariolo, D.A.: Phys. Lett. A **185**, 262 (1993)
5. Frank, T.D., Daffertshofer, A.: Phys. A **292**, 392 (2001)
6. Ito, N., Tsallis, C.: Nuovo Cim. **11**, 907 (1989)
7. Andrade, R.F.S.: Phys. A **175**, 285 (1991)
8. Andrade, R.F.S.: Phys. A **203**, 486 (1994)
9. Mariz, A.M.: Phys. Lett. A **165**, 409 (1992)
10. Ramshaw, J.D.: Phys. Lett. A **175**, 169 (1993)
11. Ramshaw, J.D.: Phys. Lett. A **175**, 171 (1993)
12. Plastino, A., Plastino, A.R.: Phys. Lett. A **177**, 177 (1993)
13. Büyükkılıç, F., Demirhan, D.: Phys. Lett. A **181**, 24 (1993)
14. Büyükkılıç, F., Demirhan, D., Güleç, A.: Phys. Lett. A **197**, 209 (1995)
15. Büyükkılıç, F., Demirhan, D.: Z. Phys. B **99**, 137 (1995)
16. Tırnaklı, U., Tsallis, C., Lyra, M.L.: Phys. Rev. E **65**, 036207 (2002)
17. Tırnaklı, U.: Phys. Rev. E **66**, 066212 (2002)
18. Rajagopal, A.K.: Phys. Rev. Lett. **74**, 1048 (1995)
19. Abe, S., Bağcı, G.B.: Phys. Rev. E **71**, 016139 (2005)

20. Bağcı, G.B., Arda, A., Sever, R.: Int. J. Mod. Phys. B **20**, 2085 (2006)
21. Lenzi, E.K., Mendes, R.S., da Silva, L.R.: Phys. A **280**, 337 (2000)
22. Parvan, A.S., Biró, T.S.: Phys. Lett. A **340**, 375 (20005)
23. Masi, M.: Phys. Lett. A **383**, 217 (2005)
24. Frank, T.D., Plastino, A.R.: Eur. Phys. J. B **30**, 543 (2002)
25. Frank, T.D., Daffertshofer, A.: Phys. A **285**, 351 (2000)
26. Sharma, B.D., Mittal, D.P.: J. Math. Sci. **10**, 28 (1975)
27. Aktürk, E., Bağcı, G.B., Sever, R.: Is Sharma-Mittal entropy really a step beyond Tsallis and Rényi entropies? arXiv:cond-mat/0703277 (2007)
28. Bialynicki-Birula, I., Mycielski, J.: Commun. Math. Phys. **44**, 129 (1975)
29. Faris, W.G.: J. Math. Phys. **19**, 461 (1978)
30. Yáñez, R.J., Assche, W.V., Dehesa, J.S.: Phys. Rev. A **50**, 3065 (1994)
31. Dehesa, J.S., Assche, W.V., Yáñez, R.J.: Methods Appl. Anal. **4**, 91 (1997)
32. Rajagopal, A.K.: Phys. Lett. A **205**, 32 (1995)
33. Bialynicki-Birula, I.: Phys. Rev. A **74**, 052101 (2006)
34. Greenberger, D.M.: J. Math. Phys. **20**, 762 (1979)
35. Abe, S., Martinez, S., Pennini, F., Plastino, A.: Phys. Lett. A **295**, 74 (2002)
36. Atre, R., Kumar, A., Kumar, C.N., Panigrahi, P.K.: Phys. Rev. A **69**, 052107 (2004)